

Trions in $1 + 1$ dimensions

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Abstract

We consider an Abelian BF-Higgs theory with $N_F = 2$ Higgs fields in $1 + 1$ dimensions. We derive a new BPS-like bound and find topological solitons with tri-charges (topological charge, Q -charge and electric charge). We call them “trions.”

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Introduction

Topological solitons are stable solutions of classical equations of motion in gauge theories and non-linear sigma models. The instantons in $4 + 1$ dimensions and the monopoles in $3 + 1$ dimensions are both well-known important solitons in the non-Abelian gauge theories. There are also solitons with di-charges. Electrically charged instantons are called dyonic instantons [1] and also monopoles with electric charge Q_e are called dyons. Interestingly, there are certain relations between solitons in higher dimensions and lower dimensions. For example, it is known that the instantons/monopoles have many properties in common with the lumps/kinks in $2 + 1/1 + 1$ dimensions. A counterpart of dyonic instanton/dyon is called Q -lump/ Q -kink [2, 3, 4] which has a conserved Noether charge Q , so-called Q -charge, instead of electric charges. Mass formulas of the dyonic instantons [1] and Q -lumps [2] are indeed quite similar as

$$M_{\text{DI}} = |Q_{\text{I}}| + |Q_e|, \quad M_{\text{QL}} = |Q_{\text{L}}| + |Q|, \quad (1)$$

and that for the dyons and Q -kinks [3, 4] are given by

$$M_{\text{Dy}} = \sqrt{Q_m^2 + Q_e^2}, \quad M_{\text{QK}} = \sqrt{Q_{\text{K}}^2 + Q^2}. \quad (2)$$

Here $Q_{\text{I}}, Q_m, Q_{\text{L}}$ and Q_{K} stand for topological charges. More direct relations can be seen when the monopoles are put into the Higgs phase. In the Higgs phase monopoles are confined and attached to non-Abelian vortices. In an effective world-sheet theory in $1 + 1$ dimensions, the monopoles are indeed identified with kinks [5, 6]. Similarly, instantons in the Higgs phase are identified with lumps [7] inside the vortex. This way, in field theories, usually topological solitons have only two charges.

Linear BPS solitons with multiple charges were found in $2 + 1$ dimensions [8]. Especially, the field theoretic supertube appears when its tension is cancelled by a linear momentum density [8]. Recently, a topological soliton with tri-charges was found in $2 + 1$ dimensional Yang-Mills-Chern-Simons-Higgs theory. It is called the dyonic non-Abelian vortex [9]. The topological vortices in Chern-Simons theories have electric charges [10, 11, 12, 13]. The dyonic non-Abelian vortex [9] has not only the topological and electric charges but also Q -charges.

Inspired by the field theoretical supertube and dyonic non-Abelian vortex, we explore topological point-like solitons with tri-charges in $1 + 1$ dimensions in this paper. We will consider $1 + 1$ dimensional BF-Higgs theory which can be obtained by dimensional reduction from the

Chern-Simons-Higgs theory in $2 + 1$ dimensions. The topological/non-topological kink solutions were found in a similar BF-Higgs theory with a Higgs field [14]. In this paper we consider $N_F = 2$ Higgs fields. We derive a new BPS-like bound and find topological solitons which have tri-charges, topological, electric and Q -charges. We will call them “trions” in $1 + 1$ dimensions.

Trions in BF-Higgs Theory

We start with the so-called BF theory coupled with a real scalar field N and $N_F = 2$ Higgs fields ϕ_1, ϕ_2 in $(1 + 1)$ dimensions with metric $\eta_{\mu\nu} = \text{diag}(+, -)$

$$\mathcal{L}_{\text{BF}} = -\kappa N F_{01} + \mathcal{D}_\mu \phi_1 (\mathcal{D}^\mu \phi_1)^* + \mathcal{D}_\mu \phi_2 (\mathcal{D}^\mu \phi_2)^* - V_{\text{BF}}, \quad (3)$$

$$V_{\text{BF}} = \sum_{i=1,2} \left[(N - n_i)^2 |\phi_i|^2 + \left(\frac{1}{\kappa} \right)^2 (|\phi_1|^2 + |\phi_2|^2 - v_i^2)^2 |\phi_i|^2 \right], \quad (4)$$

where κ is the BF coupling constant and n_i is the Higgs mass. We also introduced scalar coupling constants v_i^2 . The covariant derivative is $\mathcal{D}_\mu \phi_i = \partial_\mu \phi_i + i A_\mu \phi_i$ and the field strength is $F_{01} = \partial_0 A_1 - \partial_1 A_0$. Through out this paper, we assume $\kappa > 0$ and $v_i^2 > 0$. We also consider non-degenerate parameters, $v_1 \neq v_2$ and/or $n_1 \neq n_2$. The mass dimensions of the fields and parameters are $[\phi_i] = [v_i] = M^0$, $[A_\mu] = [N] = M^1$ and $[\kappa] = M^{-1}$. Note that the BF-Higgs theory can be obtained by a suitable dimensional reduction from the Abelian Chern-Simons theory in $2 + 1$ dimensions [14].

For later convenience, let us define a gauge invariant object

$$M \equiv \frac{1}{\kappa} (|\phi_1|^2 + |\phi_2|^2). \quad (5)$$

Then the scalar potential can be rewritten in a simple form

$$V_{\text{BF}} = \sum_{i=1,2} [(N - n_i)^2 |\phi_i|^2 + (M - m_i)^2 |\phi_i|^2], \quad (6)$$

with $m_i \equiv v_i^2/\kappa$. Classical vacua correspond to $V_{\text{BF}} = 0$. It follows that

$$\text{Higgs } \langle 1 \rangle : (N, M, \phi_1, \phi_2) = (n_1, m_1, v_1, 0), \quad (7)$$

$$\text{Higgs } \langle 2 \rangle : (N, M, \phi_1, \phi_2) = (n_2, m_2, 0, v_2), \quad (8)$$

$$\text{Coulomb } \langle 0 \rangle : (N, M, \phi_1, \phi_2) = (\mathbb{R}, 0, 0, 0). \quad (9)$$

The $U(1)$ gauge symmetry is broken in the Higgs vacua and is unbroken in the Coulomb vacua. The N and M are sufficient to specify the above vacua, so we consider the NM plane. The

Higgs vacua exist only on the upper half NM plane and the Coulomb vacua are any points on the horizontal axes ($M = 0$).

The scalar field N is a Lagrange multiplier. Its field equation gives a constraint

$$F_{01} = -\frac{2}{\kappa} \sum_{i=1,2} (N - n_i) |\phi_i|^2 \quad (10)$$

which determines distribution of the electric field.

The Bogomolnyi completion of the energy density is performed in a simple fashion by making use of M as

$$\begin{aligned} \mathcal{H} = & \sum_{i=1,2} |\mathcal{D}_1 \phi_i + [(M - m_i) \cos \alpha - (N - n_i) \sin \alpha] \phi_i|^2 \\ & + \sum_{i=1,2} |\mathcal{D}_0 \phi_i - i [(M - m_i) \sin \alpha + (N - n_i) \cos \alpha] \phi_i|^2 \\ & + \partial_1 \left[\frac{\kappa}{2} (N^2 - M^2) + \sum_{i=1,2} n_i J_{0i} + \sum_{i=1,2} m_i |\phi_i|^2 \right] \cos \alpha \\ & + \partial_1 \left[\kappa M N + \sum_{i=1,2} m_i J_{0i} - \sum_{i=1,2} n_i |\phi_i|^2 \right] \sin \alpha, \end{aligned} \quad (11)$$

where α is an arbitrary angle and we have used the Gauss' law

$$\kappa \partial_1 N + \sum_{i=1,2} J_{0i} = 0, \quad J_{0i} \equiv -i (\phi_i \mathcal{D}_0 \phi_i^* - \mathcal{D}_0 \phi_i \phi_i^*). \quad (12)$$

The similar Bogomolnyi completion for the $N_F = 1$ case with $m_1 = v^2$ and $n_1 = 0$ was found in [14, 8]. Now we introduce topological charges

$$T_X \equiv \frac{\kappa}{2} \int dx \partial_1 (X^2), \quad (X = M, N). \quad (13)$$

We also introduce the electric charge and the Noether charge so-called Q -charge

$$Q_e \equiv \int dx (J_{01} + J_{02}), \quad Q \equiv \int dx \frac{J_{01} - J_{02}}{2}. \quad (14)$$

The electric charge is expressed by the Gauss' law as

$$Q_e = -\kappa [N]_{x=-\infty}^{x=+\infty} \equiv -\kappa \delta N. \quad (15)$$

Thus any solitons with $\delta N \neq 0$ have necessarily non zero electric charge.

From Eq. (11), we can put a BPS-like bound from below on the energy $E = \int dx \mathcal{H}$ for a given charges T_M, T_N, Q_e and Q as

$$E \geq \sqrt{(T_M + T_N + \delta n Q + \bar{n} Q_e)^2 + (\delta m Q + \bar{m} Q_e)^2}, \quad \tan \alpha = \frac{\delta m Q + \bar{m} Q_e}{T_M + T_N + \delta n Q + \bar{n} Q_e}, \quad (16)$$

with $\bar{m} \equiv (m_1 + m_2)/2$ and $\delta m \equiv m_2 - m_1$ and the same for n_i . Here we made use of

$$\left[\sum_{i=1,2} m_i |\phi_i|^2 \right]_{-\infty}^{+\infty} = \kappa [M^2]_{-\infty}^{+\infty}, \quad \left[\sum_{i=1,2} n_i |\phi_i|^2 \right]_{-\infty}^{+\infty} = \kappa [MN]_{-\infty}^{+\infty}. \quad (17)$$

Thus the energy and Q -charge are summarized as

$$E = (T_M + T_N + \delta n Q + \bar{n} Q_e) \frac{1}{\cos \alpha}, \quad (18)$$

$$Q = \frac{(T_M + T_N) \tan \alpha - (\bar{m} - \bar{n} \tan \alpha) Q_e}{\delta m - \delta n \tan \alpha}. \quad (19)$$

We will call the kink with tricharges (topological, Q and electric charges) trion in 1+1 dimensions.

The Bogomolnyi bound is saturated when the first two lines in Eq. (11) vanish. For simplicity, let us introduce new variables

$$\begin{pmatrix} \hat{M} \\ \hat{N} \end{pmatrix} = U_\alpha \begin{pmatrix} M \\ N \end{pmatrix}, \quad \begin{pmatrix} \hat{m}_i \\ \hat{n}_i \end{pmatrix} = U_\alpha \begin{pmatrix} m_i \\ n_i \end{pmatrix}, \quad U_\alpha \equiv \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}. \quad (20)$$

Then we are left with the self-dual equations

$$\mathcal{D}_1 \phi_i = -(\hat{M} - \hat{m}_i) \phi_i, \quad (21)$$

$$\mathcal{D}_0 \phi_i = i(\hat{N} - \hat{n}_i) \phi_i. \quad (22)$$

Eq. (22) is solved by

$$A_0 = \hat{N}, \quad \phi_i(t, x) = e^{-i\hat{n}_i t} \tilde{\phi}_i(x). \quad (23)$$

Let us rewrite the fields by

$$\tilde{\phi}_i = e^{-\frac{\psi(x)}{2}} \phi_{0i}(x) e^{\hat{m}_i x}, \quad \frac{\psi'}{2} = \hat{M} + i A_1. \quad (24)$$

Note that there is an equivalence relation

$$\left(\phi_{0i}, e^{\frac{\psi}{2}} \right) \sim \lambda \left(\phi_{0i}, e^{\frac{\psi}{2}} \right), \quad \lambda \in \mathbb{C}^*. \quad (25)$$

This does not affect the physical fields in Eq. (24). Plugging these into Eq. (21), we get $\partial_1 \phi_{0i} = 0$. Thus ϕ_{0i} is nothing but an integration constant, namely a zero mode of the solution. We call it

the moduli matrix by analogy with the domain wall solutions in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory [15, 16]. The last task is to determine ψ for a given set of $N_F = 2$ constants $\{\phi_{0i}\}$. The Gauss' law (12) and the definition (5) lead to

$$\kappa N' = - \sum_{i=1,2} J_{0i} = 2 \sum_{i=1,2} (\hat{N} - \hat{n}_i) |\phi_i|^2, \quad (26)$$

$$\kappa M' = \sum_{i=1,2} (|\phi_i|^2)' = -2 \sum_{i=1,2} (\hat{M} - \hat{m}_i) |\phi_i|^2. \quad (27)$$

This can be put together in the following single equation

$$\kappa \hat{M}' = -2 \sum_{i=1,2} (M - m_i) |\phi_i|^2. \quad (28)$$

This is the equation for a gauge singlet object

$$\omega \equiv \frac{\psi + \psi^*}{2}, \quad \omega' = 2\hat{M}. \quad (29)$$

It follows that

$$\frac{\kappa^2}{4} \omega'' = \sum_{i=1,2} \left(m_i \kappa - \sum_{j=1,2} e^{-\omega + 2\hat{m}_j x} |\phi_{0j}|^2 \right) e^{-\omega + 2\hat{m}_i x} |\phi_{0i}|^2. \quad (30)$$

We call this the master equation again by analogy with the domain walls [15, 16].

Let us study the vacuum configurations with respect to the moduli matrix (24) with $\alpha = 0$. The i -th Higgs vacuum in Eq. (7) is given by

$$\text{Higgs } \langle i \rangle : \omega = 2m_i x + \log \frac{|a_i|^2}{m_i \kappa}, \quad \phi_{0j} = a_i \delta_{ij}. \quad (31)$$

The complex parameter a_i can be set to $a_i = 1$ by using the equivalence relation (25). This solution ω for each vacuum is called the weight of vacuum [17, 18]. The Coulomb vacua (9) is given by

$$\text{Coulomb } \langle 0 \rangle : \omega = 2m_0 x + \log a_0^2, \quad \forall \phi_{0i} = 0. \quad (32)$$

Here a_0, m_0 are arbitrary real constants. Only a_0 can be fixed by the equivalence relation (25).

There are three different topological solitons which are kinks interpolating $\langle 0 \rangle$ - $\langle 1 \rangle$, $\langle 0 \rangle$ - $\langle 2 \rangle$ and $\langle 1 \rangle$ - $\langle 2 \rangle$ vacua. Furthermore, there also exist kinks connecting two points $\langle 0 \rangle$ - $\langle 0 \rangle$ in the Coulomb vacua which are called non-topological solitons [14]. We are interested in the kink $\langle 1 \rangle$ - $\langle 2 \rangle$, namely

the trions, in this paper¹. Thus we solve the master equation (30) with the boundary condition at $x = \pm\infty$

$$\omega \rightarrow 2\hat{m}_i x + \log \frac{|a_i|^2}{m_i \kappa}. \quad (33)$$

The moduli parameter of the trion solution $\langle 1 \rangle$ - $\langle 2 \rangle$ is easily read from the moduli matrix

$$(\phi_{01}, \phi_{02}) = (a_1, a_2), \quad a_1, a_2 \in \mathbb{C}^*. \quad (34)$$

One of a_i can be fixed by the equivalence relation (25), so that the moduli space of the trion $\langle 1 \rangle$ - $\langle 2 \rangle$ is determined as

$$\mathcal{M}^{(1)-(2)} = \mathbb{C}^* \simeq \mathbb{R} \times S^1. \quad (35)$$

Here \mathbb{R} is the position which is related to the broken translational symmetry while S^1 is the Nambu-Goldstone mode associated with the broken global $U(1)$ symmetry. The position can be estimated without solving Eq. (30) just by equating two weights of vacua [19] as

$$x \simeq \frac{1}{\hat{m}_1 - \hat{m}_2} \left(\log \frac{|a_2|}{|a_1|} - \frac{1}{2} \log \frac{m_2}{m_1} \right). \quad (36)$$

To be more concrete, let us study a specific case with the following masses

$$(m_1, n_1) = (3m, m), \quad (m_2, n_2) = (m, -m), \quad (37)$$

with a non-negative constant m . Then let us solve the master equation (30) for a particular choice of the moduli matrix

$$(\phi_{01}, \phi_{02}) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad (38)$$

and with the corresponding boundary condition (33). The trion position is estimated through Eq. (36). Note that $\hat{m}_1 - \hat{m}_2$ changes its sign as

$$\hat{m}_1 \geq \hat{m}_2 \quad \text{for} \quad \alpha \in \left[-\frac{3\pi}{4}, \frac{\pi}{4} \right], \quad (39)$$

$$\hat{m}_2 \geq \hat{m}_1 \quad \text{for} \quad \alpha \in \left[-\pi, -\frac{3\pi}{4} \right], \left[\frac{\pi}{4}, \pi \right]. \quad (40)$$

An orientation of trion depends on $\hat{m}_1 - \hat{m}_2$. We will call the solutions for $\hat{m}_1 - \hat{m}_2 > 0$ trion while those for $\hat{m}_1 - \hat{m}_2 < 0$ anti-trion. In Fig. 1, we show two solutions; one is the trion with

¹ The kinks $\langle 0 \rangle$ - $\langle 1 \rangle$ and $\langle 0 \rangle$ - $\langle 0 \rangle$ were studied in [14].

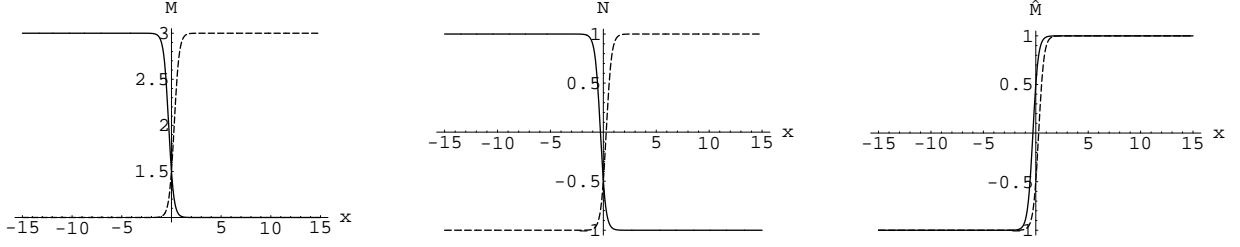


Fig. 1: The numerical solutions with $m = 1$. The solid lines are the trion ($\alpha = \pi/2$) and the dashed lines are the anti-trion ($\alpha = -\pi/2$).

$\alpha = \pi/2$ and the other is the anti-trion with $\alpha = -\pi/2$. Note that the orientation of \hat{M} is always same, see the right-most panel of Fig. 1. Because the electric charge depends on δN in Eq. (15), the trion and anti-trion have the electric charges with opposite sign. The charges are explicitly given by

$$T_M = 4\kappa m^2, \quad T_N = 0, \quad Q_e = \pm 2m\kappa, \quad (41)$$

where the upper sign is for trion and the lower for anti-trion. The Q -charge can be read from Eq. (19).

In the original Lagrangian (4), we can add an θ -term

$$\mathcal{L}_\theta = \theta F_{01}. \quad (42)$$

However, the θ -term can be absorbed in the BF term by shifting N and n_i as $N \rightarrow N + \theta/\kappa$ and $n_i \rightarrow n_i + \theta/\kappa$. Since $T_N + \bar{n}Q_e$ is invariant under the above shift, the θ -term does not play any role classically for the trions. In other words, the overall shift of n_i is unphysical.

On the contrary a shift $m_i \rightarrow m_i + v^2/\kappa$ (a non-negative constant v^2) is physically non-trivial. Then the energy is changed as

$$E = \sqrt{(T_M + \mathcal{T}_M + T_N + \delta nQ + \bar{n}Q_e)^2 + (-\mathcal{T}_N + \delta mQ + \bar{m}Q_e)^2}, \quad (43)$$

with another type of topological charges

$$\mathcal{T}_X = v^2 \int dx \partial_1 X, \quad (X = M, N). \quad (44)$$

Clearly, the above mass shift can be think of as the shift $M \rightarrow M - v^2/\kappa$. The definition of M is slightly changed from Eq. (5) as

$$\kappa M \equiv |\phi_1|^2 + |\phi_2|^2 - v^2. \quad (45)$$

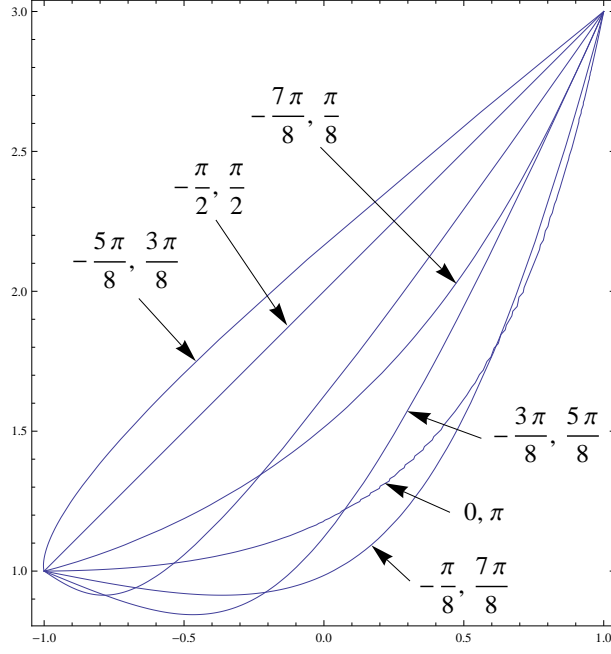


Fig. 2: The trajectories on the NM plane. Trions ($\alpha = 3\pi/8, \pi/2, 5\pi/8, 3\pi/4, 7\pi/8, \pi$) run from $(m, 3m)$ to $(-m, m)$ while anti-trions ($\alpha = -5\pi/8, -\pi/2, -3\pi/8, -\pi/4, 0$) run from $(-m, m)$ to $(m, 3m)$ with $m = 1$.

This redefinition is useful when we compare the trions and the Q -kinks [3, 4]. To this end, let us take $\kappa \rightarrow 0$ limit with v^2 and m_i being fixed ($v_i^2 = \kappa m_i \rightarrow 0$). In the limit, Eq. (45) becomes a constraint on ϕ_1, ϕ_2 which forces ϕ_1, ϕ_2 take the value in S^3 . Taking the $U(1)$ gauge symmetry into account, the BF theory with $\kappa = 0$ is a massive non-linear sigma model whose target space is $\mathbb{CP}^1 \simeq S^3/S^1$. The parameter v^2 is related to the radius of \mathbb{CP}^1 . Thus the trions in the BF theory continuously go to the Q -kinks in the \mathbb{CP}^1 model. For the Q -kinks, the master equation (30) is no longer differential equation, so that we can analytically solve it [15, 16] as

$$\omega = \log \left(v^{-2} \sum_{i=1,2} e^{2\hat{m}_i x} |\phi_{0i}|^2 \right), \quad (46)$$

and M, N are also analytically obtained as

$$M = \frac{\sum_{i=1,2} m_i e^{-\omega+2\hat{m}_i x} |\phi_{0i}|^2}{\sum_{i=1,2} e^{-\omega+2\hat{m}_i x} |\phi_{0i}|^2}, \quad N = \frac{\sum_{i=1,2} n_i e^{-\omega+2\hat{m}_i x} |\phi_{0i}|^2}{\sum_{i=1,2} e^{-\omega+2\hat{m}_i x} |\phi_{0i}|^2}. \quad (47)$$

We are ready to compare the Q -kinks and the trions. First, the Q -kinks are electrically neutral while the trions are electrically charged. Its mass is obtained from Eq. (43)

$$E_Q = \sqrt{(\mathcal{T}_M + \delta n Q)^2 + (-\mathcal{T}_N + \delta m Q)^2}. \quad (48)$$

$\mathcal{T}_{M,N}$ is the topological charge of $\mathbb{C}P^N$ kinks. Second, trajectories of Q -kinks on the NM plane are always straight segments between (n_1, m_1) and (n_2, m_2) . On the other hand, the trajectories of trions are not always straight. Generally, they are curved segments connecting (n_1, m_1) and (n_2, m_2) . We show several trajectories of trions for the masses choice (37) in Fig. 2. The trions only for $\alpha = \pm\pi/2$ are straight segments while all the others are curved trajectories. Thus the trions which we have found in this letter are quite different from the Q -kinks.

Concluding Remarks

We find new topological solitons in the $1 + 1$ dimensional BF Higgs theory. They have the topological charge, Q -charge and electric charge. Hence we call them the trions. We have derived the BPS-like energy bound and solved the self-dual equations. We also have found all the zero modes of the single trion. We have found that the trions are quite different from the ordinary Q -kinks which are electrically neutral.

There are several directions to study the trions. It was pointed out that the dyons and the Q -kinks in two dimensions share many similar properties [3, 4]. If we pursue the similarity for trions, we may expect existence of new topological solitons with tri-charges in four dimensions. One possible scenario is to seek them in the Higgs phase. Recently, a confined monopole in the Higgs phase was identified with a kink inside the non-Abelian vortex in $\mathcal{N} = 2$ supersymmetric gauge theory in four dimensions [5]. If the BF coupling is induced by radiative quantum effects on the effective vortex world-sheet theory, we would be able to construct the trions inside the vortex. From the four dimensional view point, it is the topological soliton with the tri-charge confined inside the flux tube. It is also interesting to find multiple trions. In our model there exists only one trion since we have only two Higgs vacua. In models with more scalar fields $N_F > 2$, we have N_F Higgs vacua and there may exist $N_F - 1$ multiple trions in such models. Scattering of multiple trions is also an interesting problem. Moreover, it may be interesting to explore trions in non-relativistic BF theories and in non-Abelian BF theories.

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